THE VALUATION DIFFERENCE RANK OF AN ORDERED DIFFERENCE FIELD

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Abstract. There are several equivalent characterizations of the valuation rank of an ordered field (endowed with its natural valuation). In this paper, we extend the theory to the case of an ordered difference field and introduce the notion of *difference rank*. We characterize the difference rank as the quotient modulo the equivalence relation naturally induced by the automorphism (which encodes its growth rate). In analogy to the theory of convex valuations, we prove that any linearly ordered set can be realized as the difference rank of an ordered difference field.

1. Introduction

The theory of real places and convex valuations is a special chapter in valuation theory; it is a basic tool in real algebraic geometry. Surveys can be found in [5], [6] and [7]. An important isomorphism invariant of an ordered field is its rank as a valued field, which has several equivalent characterizations: via the ideals of the valuation ring, the value group, or the residue field . This can be extended to ordered fields with extra structure, giving a characterization completely analogous to the above, but taking into account the corresponding induced structure on the ideals, value group, or residue field (see [2] for ordered exponential fields).

In this paper, we push this analogy to the case of an (ordered) difference field. The leading idea is to identify the difference rank of a non-archimedean ordered field as the quotient by the equivalence relation that the automorphism induces on the set of infinitely large field elements. In Section 2 we start by a key remark regarding equivalence relations defined by monotone maps on chains, and briefly review the theory of convex valuations and rank. In Section 3 we represent this invariant via equivalence relations induced by addition and multiplication on the field. This approach allows us to develop in Section 4 the notion of difference compatible valuations, introduce the difference rank, and consider in particular isometries, weak isometries and ω-increasing automorphisms. The main results of the paper are in the last Section 5: Theorem 5.2 and its Corollaries 5.4, 5.3 and 5.5.

Finally, we note that some of our notions and results generalize to the context of an arbitrary (not necessarily ordered) valued field, see [3].

2. Preliminaries on the rank and principal rank of an ordered field

We begin by the following key observation:

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Remark 2.1. Let φ be a map from a totally ordered set S into itself, and assume that φ is order preserving, i. e. $a \leq a'$ implies $\varphi(a) \leq \varphi(a')$, for all $a,\ a' \in S$. We assume further that φ has an orientation, i. e. $\varphi(a) \geq a$ for all $a \in S$ (φ is a right shift) or $\varphi(a) \leq a$ for all $a \in S$ (φ is a left shift). We set $\varphi^0(a) := a$ and $\varphi^{n+1}(a) := \varphi(\varphi^n(a))$ for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

It is then straightforward that the following defines an equivalence relation on S:

(i) If φ is a right shift, set $a \sim_\varphi a'$ if and only if there is some $n \in \mathbb{N}_0$ such that $\varphi^n(a) \geq a'$ and $\varphi^n(a') \geq a$,

(ii) If φ is a left shift, set $a \sim_\varphi a'$ if and only if there is some $n \in \mathbb{N}_0$ such that $\varphi^n(a) \leq a'$ and $\varphi^n(a') \leq a$.

The equivalence classes $[a]_\varphi$ of $\sim_\varphi$ are convex and closed under application of φ. By the convexity, the order of S induces an order on $S/{\sim_\varphi}$ such that $[a]_\varphi < [b]_\varphi$ if and only if $a' < b'$ for all $a' \in [a]_\varphi$ and $b' \in [b]_\varphi$.

On the negative cone $G^{<0}$ of an ordered abelian group G, the **archimedean equivalence** relation $\sim_\varphi$ is obtained by setting $\varphi(a) := 2a$, and v_G is the map $a \mapsto [a]_\varphi$. The order on $\Gamma := G^{<0}/\sim_\varphi$ is the one induced by the order of $G^{<0}$ as above. We call $v_G(G^{<0}) := \Gamma$ the **value set of** G. By convention we also write $v_G(G) := \Gamma$ extending the archimedean equivalence relation to the positive cone of G by setting $v_G(g) := v_G(-g)$. The natural valuation v_G on G satisfies the ultrametric triangle inequality, and in particular we have: $v_G(x + y) = \min\{v_G(x), v_G(y)\}$ if $\mathrm{sign}(x) = \mathrm{sign}(y)$.

We gather some basic facts about valuations compatible with the order of an ordered field. Throughout this paper, K will be a non-archimedean ordered field, and v will denote its non-trivial natural valuation, that is, its valuation ring R_v is the convex hull of $\mathbb{Q}$ in K. We set: $\mathbf{P}_K := K^{>0} \setminus R_v$, $G = v(K)$ and $\Gamma = v_G(G^{<0})$. The natural valuation on K satisfies $v(x+y) = \min\{v(x), v(y)\}$ if $\mathrm{sign}(x) = \mathrm{sign}(y)$ and for all $a, b \in K\ : a \geq b > 0 \implies v(a) \leq v(b)$.

Let w be a valuation of K, with valuation ring R_w , valuation ideal I_w , value group $w(K)$ and residue field Kw. Then w is called **compatible with the order** if and only if it satisfies, for all $a, b \in K\ a \geq b > 0 \implies w(a) \leq w(b)$.

For the following characterizations of compatible valuations, see [4] Proposition 5.1, or [5] Theorem 2.3 and Proposition 2.9, or [7] Lemma 3.2.1, or [8] Lemma 7.2:

Lemma 2.2. *The following assertions are equivalent:*
1) w *is a valuation compatible with the order of* $(K, <)$,
2) R_w *is a convex subset of* $(K, <)$,
3) I_w *is a convex subset of* $(K, <)$,
4) $I_w < 1$,
5) the image of the positive cone P *of* $(K, <)$ *under the residue map* $K \ni a \mapsto aw \in Kw$ *is a positive cone* Pw *in* Kw.

A valuation compatible with the order is thus also said to be a **convex valuation**. For every convex valuation w, we denote by $\mathcal{U}_w^{>0} := \{a \in K \mid w(a) = 0 \wedge a > 0\}$ the group of **positive units** of R_w. It is a convex subgroup of the ordered multiplicative group $(K^{>0}, \cdot, 1, <)$ of positive elements of K.

Let w and w' be valuations on K. We say that w' is **finer** than w, or that w is **coarser** than w' if $R_{w'} \subsetneqq R_w$. This is equivalent to $I_w \subsetneqq I_{w'}$. By Lemma 2.2 4), if w' is convex, then w also is convex. Conversely, a convex subring containing 1 is a valuation ring, see [1] Section 2.2.2. The natural valuation v of K is the finest convex valuation (and is characterized by the fact that its residue field is archimedean). The set $\mathcal{R}$ of all valuation rings R_w of convex valuations $w \neq v$ (i. e. all corsenings of v) is totally ordered by inclusion. Its order type is called the **rank** of $(K, +, \cdot, 0, 1, <)$. For convenience, we will identify it with $\mathcal{R}$. For example, the rank of an archimedean ordered field is empty since its natural valuation is trivial. The rank of the rational function field $K = \mathbb{R}(t)$ with any order is a singleton: $\mathcal{R} = \{K\}$.

Recall that the set of all convex subgroups $G_w \neq \{0\}$ of the value group G is totally ordered by inclusion. Its order type is called the **rank** of G. Note that the rank of an ordered field (respectively of an ordered group) is an isomorphism invariant. To every convex valuation ring R_w, we associate a convex subgroup $G_w := \{v(a) \mid a \in K \wedge w(a) = 0\} = v(\mathcal{U}_w^{>0})$. We call G_w the **convex subgroup associated to** w. Note that $G_v = \{0\}$. Conversely, given a convex subgroup G_w of $v(K)$, we define $w : K \to v(K)/G_w$ by $w(a) = v(a) + G_w$. Then w is a convex valuation with $v(\mathcal{U}_w^{>0}) = G_w$ (and v is finer than w if and only if $G_w \neq \{0\}$). We call w the **convex valuation associated to** G_w. We summarize:

Lemma 2.3. *The correspondence $R_w \mapsto G_w$ is an order preserving bijection, thus $\mathcal{R}$ is (isomorphic to) the rank of G.*

We want to analyze the rank of G further and relate it to the value set Γ of G. For $G_w \neq \{0\}$ a convex subgroup, we associate $\Gamma_w := v_G(G_w^{<0})$ a non-empty final segment of Γ. Conversely, if Γ_w is a non-empty final segment of Γ, then $G_w = \{g \mid g \in G, v_G(g) \in \Gamma_w\} \cup \{0\}$ is a convex subgroup, with $\Gamma_w = v_G(G_w)$. Let us denote by Γ^{fs} the set of non-empty final segments of Γ, totally ordered by inclusion. We summarize:

Lemma 2.4. *The correspondence $G_w \mapsto \Gamma_w$ is an order preserving bijection, thus the rank of G is (isomorphic to) Γ^{fs}.*

Theorem 2.5. *The correspondence $R_w \mapsto \Gamma_w$ is an order preserving bijection, thus $\mathcal{R}$ is (isomorphic to) Γ^{fs}.*

A final segment which has a smallest element is a **principal final segment**. Let Γ^* denote the set Γ with its reversed ordering, then Γ^* is (isomorphic to) the totally ordered set of principal final segments:

Lemma 2.6. *The map from Γ to Γ^{fs} defined by $\gamma \mapsto \{\gamma' \mid \gamma' \in \Gamma, \gamma' \geq \gamma\}$ is an order reversing embedding. Its image is the set of principal final segments.*

Recall that a convex subgroup G_w of G is called **principal generated by** g, $g \in G$, if G_w is the minimal convex subgroup containing g. The **principal rank** of G is the subset of the rank of G consisting of all principal $G_w \neq \{0\}$.

Lemma 2.7. *Let $G_w \neq \{0\}$ be a convex subgroup. Then G_w is principal if and only if $v_G(G_w) = \Gamma_w$ is a principal final segment.*

Lemma 2.8. *The map $G_w \mapsto \min v_G(G_w)$ is an order reversing bijection. Thus the principal rank of G is (isomorphic to) Γ^*.*

A convex subring $R_w \neq R_v$ is a **principal convex subring generated by** a for $a \in \mathbf{P}_K$ if R_w is the smallest convex subring containing a. The **principal rank** of K is the subset $\mathcal{R}^{\mathrm{pr}}$ of $\mathcal{R}$ consisting of all principal $R_w \in \mathcal{R}$.

Theorem 2.9. *The correspondence $R_w \mapsto \Gamma_w$ is an order preserving bijection between $\mathcal{R}^{\mathrm{pr}}$ and the principal rank of G, thus $\mathcal{R}^{\mathrm{pr}}$ is (isomorphic to) Γ^*.*

3. The rank and principal rank via equivalence relations

In this section, we exploit Remark 2.1 to give an interpretation of the rank and principal rank as quotients via an appropriate equivalence relation, thereby providing alternative proofs for Theorem 2.5 and Theorem 2.9. It is precisely this approach that we will generalize to the difference rank in the next sections.

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbf{P}_K \arrow[r, "\varphi"] \arrow[d, "v"] \arrow[dr, phantom, "///"] & \mathbf{P}_K \arrow[d, "v"] \\
G^{<0} \arrow[r, "\varphi_G"] \arrow[d, "v_G"] \arrow[dr, phantom, "///"] & G^{<0} \arrow[d, "v_G"] \\
v_G(G) \arrow[r, "\varphi_\Gamma"] & v_G(G)
\end{tikzcd}$$

with $\varphi(a) \;:=\; a^2$ for all $a \in \mathbf{P}_K$,

$\varphi_G(v(a)) \;:=\; v(\varphi(a))$ for all $a \in \mathbf{P}_K$,

that is $\varphi_G(g) = 2g$ for all $g \in G^{<0}$, and

$\varphi_\Gamma(v_G(g)) \;:=\; v_G(\varphi_G(g))$ for all $g \in G^{<0}$,

that is $\varphi_\Gamma(\gamma) = \gamma$ for all $\gamma \in \Gamma$, so that φ_Γ is just the identity map.

We consider the equivalence relations associated to the monotone maps φ, φ_G and φ_Γ as in Remark 2.1, and note that $\sim_\varphi$ is just multiplicative equivalence $\sim^{\cdot}$ on $\mathbf{P}_K$, $\sim_{\varphi_G}$ just archimedean equivalence on G and $\sim_{\varphi_\Gamma}$ just equality on Γ. Suppose that $\sim_1$ and $\sim_2$ are two equivalence relations defined on the same set. Recall that $\sim_1$ is said to be **coarser** than $\sim_2$ if $\sim_2$-equivalence implies $\sim_1$-equivalence. The following straightforward observation will be useful for the proof of Theorem 3.2 below:

Lemma 3.1. *The equivalence relation $\sim_\varphi$ is coarser than the archimedean equivalence relation with respect to addition on $\mathbf{P}_K$. The equivalence classes of $\sim_\varphi$ are closed under addition and multiplication.*

We further note that

$$\varphi_G^n(v(a)) \;=\; v(\varphi^n(a)) \text{ and } \varphi_\Gamma^n(v_G(g)) \;=\; v_G(\varphi_G^n(g))$$

thus

$$a \sim_\varphi a' \text{ if and only if } v(a) \sim_{\varphi_G} v(a') \text{ if and only if } v_G(v(a)) \sim_{\varphi_\Gamma} v_G(v(a'))\,.$$

Thus we have an order reversing bijection from $\mathbf{P}_K/\sim_\varphi$ onto $\Gamma/\sim_{\varphi_\Gamma}= \Gamma$. Thus the chain $[\mathbf{P}_K/\sim_\varphi]^{\mathrm{is}}$ of initial segments of $\mathbf{P}_K/\sim_\varphi$ ordered by inclusion is isomorphic to Γ^{fs}. Theorems 2.5 and 2.9 will therefore immediately follow from the following result

Theorem 3.2. *The rank $\mathcal{R}$ is isomorphic to $[\mathbf{P}_K/\sim_\varphi]^{\mathrm{is}}$ and the principal rank $\mathcal{R}^{\mathrm{pr}}$ is isomorphic to the subset of $[\mathbf{P}_K/\sim_\varphi]^{\mathrm{is}}$ of initial segments which have a last element.*

Proof. First we note that if R_w is a convex valuation ring, then clearly $R_w^{>0} \setminus R_v^{>0}$ is an initial segment of $\mathbf{P}_K$, and moreover $[a]_{\sim_\varphi}$ is the last class in case R_w is principal generated by a. Furthermore, if R_w intersects an equivalence class $[a]_{\sim_\varphi}$ then it must contain it, since the sequence $a^n; n \in \mathbb{N}_0$ is cofinal in $[a]_{\sim_\varphi}$ and R_w is a convex subring. We conclude that $(R_w^{>0} \setminus R_v^{>0})/{\sim_\varphi}$ is an initial segment of $\mathbf{P}_K/{\sim_\varphi}$. Conversely set $\mathcal{I}_w = \{[a]_\varphi \mid a \in R_w^{>0} \setminus R_v^{>0}\}$. Given $\mathcal{I} \in [\mathbf{P}_K/{\sim_\varphi}]^{\mathrm{is}}$, we show that there is a convex valuation ring R_w such that $\mathcal{I}_w = \mathcal{I}$. Given $\mathcal{I}$, let $(\bigcup \mathcal{I})$ denote the set theoretic union of the elements of $\mathcal{I}$ and $-(\bigcup \mathcal{I})$ the set of additive inverses. Set $R_w = -(\bigcup \mathcal{I}) \cup R_v \cup (\bigcup \mathcal{I})$. We claim that R_w is the required ring. Clearly, $\mathcal{I}_w = \mathcal{I}$. Further R_w is convex (by its construction), and strictly contains R_v. We leave it to the reader, using Lemma 3.1, to verify that R_w is a ring, and that R_w is principal generated by a if $[a]_{\sim_\varphi}$ is the last element of $\mathcal{I}$. $\square$

Note that the principal rank determines the rank, that is if ordered fields have (isomorphic) principal ranks, then they have (isomorphic) ranks.

4. The difference analogue of the rank

In this section, we develop a difference analogue of what has been reviewed above. That is, we develop a theory of difference compatible valuations, in analogy to the theory of convex valuations. The automorphism will play the role that multiplication plays in the previous case.

Let σ be an order preserving automorphism of K. We assume $\sigma \neq$ identity, i.e. $(K, <, \sigma)$ is a non-trivial ordered difference field. Note that σ satisfies for all $a, b \in K$: $v(a) \leq v(b)$ if and only if $v(\sigma(a)) \leq v(\sigma(b))$ and thus induces an order preserving automorphism σ_G and σ_Γ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{P}_K \arrow[r, "\sigma"] \arrow[d, "v"] \arrow[dr, phantom, "///"] & \mathbf{P}_K \arrow[d, "v"] \\
G^{<0} \arrow[r, "\sigma_G"] \arrow[d, "v_G"] \arrow[dr, phantom, "///"] & G^{<0} \arrow[d, "v_G"] \\
v_G(G) \arrow[r, "\sigma_\Gamma"] & v_G(G)
\end{tikzcd}$$

with $\sigma_G(v(a)) := v(\sigma(a))$ for all $a \in \mathbf{P}_K$,

and

$\sigma_\Gamma(v_G(g)) := v_G(\sigma_G(g))$ for all $g \in G^{<0}$.

Now let w be a convex valuation on K. Say w is **σ-compatible** if for all $a, b \in K$: $w(a) \leq w(b)$ if and only if $w(\sigma(a)) \leq w(\sigma(b))$.

The subset $\mathcal{R}_\sigma := \{ R_w \in \mathcal{R} \,;\, R_w$ is σ- compatible $\}$ is the **σ-rank** of $(K, <, \sigma)$. Similarly, the subset of all convex subgroups $G_w \neq \{0\}$ such that $\sigma_G(G_w) = G_w$ (σ_G-invariant) is the **σ-rank** of G. Finally, we denote by σ_Γ-Γ^{fs} the subset of final segments Γ_w such that $\sigma_\Gamma(\Gamma_w) = \Gamma_w$.

The following analogues of Lemmas 2.2, 2.3 and 2.4 are verified by straightforward computations, using basic properties of valuations rings on the one hand and of automorphisms on the other (e.g. $\sigma(A) \subseteq B$ if and only if $A \subseteq \sigma^{-1}(B)$ and $\sigma(A) \subseteq B$ if and only if $\sigma(-A) \subseteq -B$):

Lemma 4.1. *The following assertions are equivalent for a convex valuation w :*
1) w is σ–compatible
2) w is σ^{-1}–compatible
3) $\sigma(R_w) = R_w$
4) $\sigma(I_w) = I_w$
5) $\sigma(R_w^{>0} \setminus R_v^{>0}) = R_w^{>0} \setminus R_v^{>0}$
6) the map $Kw \to Kw$ defined by $aw \mapsto \sigma(a)w$ is well-defined and is an order preserving automorphism of Kw (with the ordering induced by Pw).

We shall call R_w σ-compatible if any of the above equivalent conditions holds.

Lemma 4.2. *The correspondence $R_w \mapsto G_w$ is an order preserving bijection from $\mathcal{R}_\sigma$ onto the σ_G-rank of G.*

Lemma 4.3. *The correspondence $G_w \mapsto \Gamma_w$ is an order preserving bijection from the σ_G-rank of G onto σ_Γ-Γ^{fs}.*

We deduce from Lemmas 4.2 and 4.3 the following information. An automorphism σ is an **isometry** if $v(\sigma(a)) = v(a)$ for all $a \in K$, equivalently σ_G is the identity automorphism, and a **weak isometry** if σ_Γ is the identity automorphism. Every isometry is a weak isometry. Note that if Γ is a rigid chain, then σ is necessarily a weak isometry. If σ is a weak isometry, then $\sigma_\Gamma(v_G(g)) = v_G(\sigma_G(g)) = v_G(g)$, thus g is archimedean equivalent to $\sigma_G(g)$ for all g, and so every convex subgroup is σ_G-invariant.

Corollary 4.4. *If σ is a weak isometry, then $\mathcal{R}_\sigma = \mathcal{R}$.*

Corollary 4.5. *The correspondence $R_w \mapsto \min \Gamma_w$ is an order (reversing) isomorphism from $\mathcal{R}_\sigma \cap \mathcal{R}^{\mathrm{pr}}$ onto the chain $\{\gamma \,;\, \sigma_\Gamma(\gamma) = \gamma\}$ of fixed points of σ_Γ.*

At the other extreme σ is said to be **ω-increasing** if $\sigma(a) > a^n$ for all $n \in \mathbb{N}_0$ and all $a \in \mathbf{P}_K$.

Remark 4.6. Note that σ is ω-increasing if and only if σ_Γ is a **strict left shift**, that is, $\sigma_\Gamma(\gamma) < \gamma$ for all $\gamma \in \Gamma$. Thus if σ ω-increasing then σ_Γ has no fixed points.

Corollary 4.7. *If σ is ω-increasing then $\mathcal{R}_\sigma \cap \mathcal{R}^{\mathrm{pr}}$ is empty.*

Recall that the **Hahn group** over the chain γ and components $\mathbb{R}$, denoted $\mathbf{H}_\Gamma\mathbb{R}$, is the totally ordered abelian group whose elements are formal sums $g := \sum g_\gamma 1_\gamma$,

with well-ordered support $g := \{\gamma\ ;\ g_\gamma \neq 0\}$. Addition is pointwise and the order lexicographic. Similarly, the the field of **generalized power series** over the ordered abelian group G (or **Hahn field** over G), denoted $\mathbb{R}((G))$, is the totally ordered field whose elements are formal series $s := \sum s_g t^g$, with well-ordered support $s := \{g\ ;\ s_g \neq 0\}$. Addition is pointwise, multiplication is given by the usual convolution formula, and the order is lexicographic.

Lemma 4.8. *Any order preserving automorphism σ_Γ of the chain Γ lifts to an order preserving automorphism σ_G of the Hahn group G over Γ, and σ_G lifts in turn to an order preserving automorphism σ of the Hahn field over G.*

Proof. Set $\sigma_G(\sum g_\gamma 1_\gamma) := \sum g_\gamma 1_{\sigma_\Gamma(\gamma)}$ and $\sigma(\sum s_g t^g) := \sum s_\gamma t^{\sigma_G(g)}$. □

Corollary 4.9. *Given any order type τ there exists a maximally valued non-trivial ordered difference field $(K, <, \sigma)$, such that the order type of $\mathcal{R}_\sigma \cap \mathcal{R}^{\mathrm{pr}}$ is τ.*

Proof. Set $\mu := \tau^*$, and consider e.g. the linear ordering $\Gamma := \sum_\mu \mathbb{Q}^{\geq 0}$, that is, the concatenation of μ copies of the non-negative rationals. Fix a non-trivial order automorphism η of $\mathbb{Q}^{>0}$. Define σ_Γ to be the uniquely defined order automorphism of Γ fixing every $0 \in \mathbb{Q}^{\geq 0}$ in every copy and extending η otherwise on every copy. Set e.g. $G := \mathbf{H}_\Gamma \mathbb{R}$. Then σ_Γ lifts canonically to σ_G on G. Now set e.g. $K := \mathbb{R}((G))$. Again σ_G lifts canonically to an order automorphism of K, our required σ. □

In the next section, we will exploit appropriate equivalence relations to define the principal difference rank and construct difference fields of arbitrary difference rank.

5. The principal σ-rank

Our aim now is to state and prove the analogues to Theorems 3.2, 2.5 and 2.9. However, scrutinizing the proof of Theorem 3.2 we quickly realize that in order to obtain an analogue of Lemma 3.1 (which is essential for the arguments), we need further condition on σ. *Thus from now on we will assume that $\sigma(a) \geq a^2$ for all $a \in \mathbf{P}_K$.* It follows by induction that $\sigma^n(a) \geq a^{2^n}$. Thus given $n \in \mathbb{N}_0$, there exists $l \in \mathbb{N}_0$ such that $\sigma^l(a) \geq a^n$. Note that our condition on σ is fulfilled for ω-increasing automorphism.

A convex subring $R_w \neq R_v$ is **σ-principal generated by** a for $a \in \mathbf{P}_K$ if R_w is the smallest convex σ-compatible subring containing a. The **σ-principal rank** of K is the subset $\mathcal{R}_\sigma^{\mathrm{pr}}$ of $\mathcal{R}_\sigma$ consisting of all σ-principal $R_w \in \mathcal{R}$.

The maps σ, σ_G and σ_Γ are order preserving and we can define the corresponding equivalence relations $\sim_\sigma$, $\sim_{\sigma_G}$ and $\sim_{\sigma_\Gamma}$. As before we have

$$a \sim_\sigma a' \text{ if and only if } v(a) \sim_{\sigma_G} v(a') \text{ if and only if } v_G(v(a)) \sim_{\sigma_\Gamma} v_G(v(a'))\ .$$

Thus we have an order reversing bijection from $\mathbf{P}_K/\sim_\sigma$ onto $\Gamma/\sim_{\sigma_\Gamma}$. Thus the chain $[\mathbf{P}_K/\sim_\sigma]^{\mathrm{is}}$ of initial segments of $\mathbf{P}_K/\sim_\sigma$ ordered by inclusion is isomorphic to $[\Gamma/\sim_{\sigma_\Gamma}]^{\mathrm{fs}}$.

Lemma 5.1. *(i) The equivalence relation $\sim_\sigma$ is coarser than the archimedean equivalence relations with respect to addition and multiplication on $\mathbf{P}_K$.*

(ii) The equivalence classes of $\sim_\sigma$ are thus closed under addition, multiplication and σ.

Proof. If a is archimedean equivalent to b then $v(a) = v(b)$ so $v(a) \sim_{\sigma_G} v(b)$ certainly and therefore $a \sim_\sigma b$. If a is multiplicatively equivalent to b so that $a^n \geq b$ and $b^n \geq a$ for some $n \in \mathbb{N}_0$, then choose l large enough so that $\sigma^l(a) \geq a^n$ and $\sigma^l(b) \geq b^n$. Clearly, the condition on σ implies that $a \sim_\sigma \sigma(a)$. Recall that the natural valuation on K satisfies $v(x+y) = \min\{v(x), v(y)\}$ if $\mathrm{sign}(x) = \mathrm{sign}(y)$. One easily deduces from this fact and (i) that the equivalence classes of σ are closed under addition. Similarly, the natural valuation v_G on G satisfies $v_G(x+y) = \min\{v_G(x), v_G(y)\}$ if $\mathrm{sign}(x) = \mathrm{sign}(y)$. Again one easily deduces from this fact and (i) that the equivalence classes of σ are closed under multiplication. □

We can now prove:

Theorem 5.2. *The σ-rank $\mathcal{R}_\sigma$ is isomorphic to $[\mathbf{P}_K/\sim_\sigma]^{\mathrm{is}}$ and the principal σ-rank $\mathcal{R}_\sigma^{\mathrm{pr}}$ is isomorphic to the subset of $[\mathbf{P}_K/\sim_\sigma]^{\mathrm{is}}$ of initial segments which have a last element.*

Proof. First we note that if R_w is a convex σ-compatible valuation ring, then clearly $R_w^{>0} \backslash R_v^{>0}$ is an initial segment of $\mathbf{P}_K$. Furthermore, if R_w intersects a σ- equivalence class $[a]_{\sim_\sigma}$ then it must contain it, since the sequence $\sigma(a)^n; n \in \mathbb{N}_0$ is cofinal in $[a]_{\sim_\sigma}$ and R_w is a convex subring. We conclude that $(R_w^{>0} \setminus R_v^{>0})/\sim_\sigma$ is an initial segment of $\mathbf{P}_K/\sim_\sigma$ and moreover $[a]_{\sim_\sigma}$ is the last class in case R_w is σ- principal generated by a. Conversely set $\mathcal{I}_w = \{[a]_\sigma \mid a \in R_w^{>0} \setminus R_v^{>0}\}$. Given $\mathcal{I} \in [\mathbf{P}_K/\sim_\sigma]^{\mathrm{is}}$, we show that there is a σ-compatible convex valuation ring R_w such that $\mathcal{I}_w = \mathcal{I}$. Given $\mathcal{I}$, let $(\bigcup \mathcal{I})$ denote the set theoretic union of the elements of $\mathcal{I}$ and $-(\bigcup \mathcal{I})$ the set of additive inverses. Set $R_w = -(\bigcup \mathcal{I}) \cup R_v \cup (\bigcup \mathcal{I})$. We claim that R_w is the required ring. Clearly, $\mathcal{I}_w = \mathcal{I}$. Further R_w is convex (by its construction), and strictly contains R_v. We leave it to the reader, using Lemma 5.1, to verify that R_w is a σ-compatible subring, and that R_w is σ-principal generated by a if $[a]_{\sim\sigma}$ is the last element of $\mathcal{I}$. □

Corollary 5.3. *$\mathcal{R}_\sigma$ is (isomorphic to) $(\Gamma/\sim_{\sigma_\Gamma})^{\mathrm{fs}}$.*

Corollary 5.4. *$\mathcal{R}_\sigma^{\mathrm{pr}}$ is (isomorphic to) $(\Gamma/\sim_{\sigma_\Gamma})^*$.*

We now can ω-increasing automorphisms of arbitrary principal difference rank:

Corollary 5.5. *Given any order type τ there exists a maximally valued ordered field endowed with an ω-increasing automorphism of principal difference rank τ.*

Proof. Set $\mu := \tau^*$, and consider e.g. the linear ordering $\Gamma := \sum_\mu \mathbb{Q}$, that is, the concatenation of μ copies of the non-negative. Let ℓ be e.g. translation by -1 on $\mathbb{Q}$. Define σ_Γ to be the uniquely defined order automorphism of Γ extending ℓ on every copy. It is a left shift. Set e.g. $G := \mathbf{H}_\Gamma\mathbb{R}$. Then σ_Γ lifts canonically to σ_G on G. Now set e.g. $K := \mathbb{R}((G))$. Again σ_G lifts canonically to an order automorphism of K, our required σ. □

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